Some perturbation solutions in laminar boundary layer theory

Part 2. The energy equation

By HERBERT FOX AND PAUL A. LIBBY
Polytechnic Institute of Brooklyn

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Solutions for two types of problems involving the energy equation for flows with velocities described by the Blasius solution are presented. The first type arises in flows with arbitrary initial distributions of stagnation enthalpy and with surfaces downstream of the initial station either with constant wall enthalpy or with zero heat transfer. Exact solutions in these cases are obtained for constant $\rho\mu$, and Prandtl number of unity; they are given in terms of complete orthogonal sets of functions which can be used to obtain first- and higher-order corrections for the effects of variable $\rho\mu$, non-unity Prandtl number, and deviations of the velocity field from that described by the Blasius solution.

The second type of problem pertains to flows with power-law descriptions of the wall enthalpy. Again the basic solutions are obtained for Prandtl number of unity and the effect of non-unity Prandtl number is treated as a perturbation.

1. Introduction

As a continuation of the study of flows with velocities described by perturbations of the Blasius solution (Libby & Fox 1963) there is presented herein a related study of the energy equation. Because of the relation among species conservation with no gas-phase chemical reaction, element conservation and energy conservation expressed in terms of the stagnation enthalpy, the analysis can also be applied directly to problems related to heterogeneous flows. Although this applicability will not be repeatedly emphasized, it will be readily appreciated.

If the product of the mass density and viscosity coefficient ($\rho\mu$) and the Prandtl number are constant, the energy equation is a linear partial differential equation which for a known, but general, velocity field is difficult to solve. However, if the velocities are given by the Blasius solution, exact solutions can be obtained for a variety of initial and boundary conditions with the Prandtl number as the sole parameter.

If the Prandtl number is close to unity as in most gas flows so that the effects of non-unity Prandtl number can be considered as perturbations, these exact solutions, which may be considered basic, and the related perturbation solutions can be obtained once and for all and can be applied readily to obtain approximate solutions for arbitrary Prandtl number. In the same manner the influences of deviations of the velocity field from that described by the Blasius solution due either to mass transfer or to variations in the product $\rho\mu$ can be considered as perturbations to the basic solutions.

There are first presented solutions of two problems which form the basis of the perturbation analysis and which correspond to $C \equiv 1$, $\sigma \equiv 1$, and $f \equiv f_0(\eta)$; these pertain to initial profile problems wherein the distribution of stagnation enthalpy at a given streamwise station is arbitrarily specified. In one the enthalpy distribution on the wall downstream of the initial station is constant and in the second the heat transfer to the wall is zero. The solutions to these problems are shown to be given in terms of complete orthogonal sets of functions. The perturbations due to deviations of C, σ and f are next shown to be determined in terms of a Green's function and thus may be obtained by quadrature. The results of this analysis are applied to the problem of the heat transfer to a surface with a step-function distribution of wall enthalpy, of the adiabatic wall enthalpy downstream of a cooled or heated surface, and of the adiabatic wall enthalpy downstream of a porous-cooled surface.

There is presented next a reconsideration of the problem which has been treated by Chapman & Rubesin (1949) and which involves the heat transfer to a surface with a power-law distribution of the wall enthalpy. In this problem the Prandtl number is a parameter; here the deviation thereof from unity is treated as a perturbation so that the results may be readily applied to any constant, non-unity Prandtl number. Although these arbitrary values must presumably be in the neighbourhood of unity, the results are compared to more accurate calculations made with Prandtl number as a parameter and are shown to be in good agreement for $0{\cdot}5 < \sigma < 2{\cdot}0$.

As an extension of the Chapman–Rubesin problem there is considered a flow with a specified initial distribution of stagnation enthalpy and a wall enthalpy expressed as a power series in the streamwise variable. For this problem it may be convenient to express the wall enthalpy in terms of negative powers; the requisite functions are therefore presented here.

The report concludes with a discussion of several applications of the various solutions presented herein. Reference will be made throughout the discussion to pertinent related work.†

2. Initial-profile problems

The energy equation for a laminar boundary layer with a uniform external stream and a constant Prandtl number σ can be written in terms of the Levy–Lees variables η and $\tilde{s}$ and of the stagnation enthalpy ratio g [cf. Lees 1956 and Hayes & Probstein 1959] as‡

$$(Cg_\eta)_\eta + \sigma f g_\eta + 2\tilde{m}\,(\sigma - 1)\,(Cf_\eta f_{\eta\eta})_\eta - 2\tilde{s}\sigma(f_\eta g_{\tilde{s}} - f_{\tilde{s}} g_\eta) = 0. \tag{2.1}$$

† Tabulations of various functions, which were generated in this study and which may be of value in application, are available in a complete version of this paper identified as PIBAL Report No. 704, October 1963.

‡ Note that it is assumed either that the composition is uniform throughout the boundary layer or that the Lewis number associated with the diffusion coefficient of each species is unity.

The corresponding momentum equation is

$$(Cf_{\eta\eta})_\eta + ff_{\eta\eta} - 2\tilde{s}(f_\eta f_{\eta\tilde{s}} - f_s f_{\eta\eta}) = 0. \tag{2.2}$$

The analysis is started by letting

$$f(\tilde{s},\eta) = f_0(\eta) + f_1(\tilde{s},\eta), \quad g(\tilde{s},\eta) = g_0(\tilde{s},\eta) + g_1(\tilde{s},\eta)$$

and by writing (2.1) in a formal way, i.e. without approximation, as

$$\begin{aligned} g_{0_{\eta\eta}} + f_0 g_{0_\eta} - 2\tilde{s}f_0' g_{0_{\tilde{s}}} = \{ & -g_{1_{\eta\eta}} - f_0 g_{1_{\eta\eta}} + 2\tilde{s}f_0' g_{1_{\tilde{s}}} \\ & + [(1-C)\, g_{0_\eta}]_\eta - [f_1 g_{0_\eta} - 2\tilde{s}(f_{1_\eta} g_{0_{\tilde{s}}} - f_{1_s} g_{0_\eta})] + (1-\sigma)[f_0 g_{0_\eta} \\ & + 2\tilde{m}(f_0' f_0'')' - 2\tilde{s}f_0' g_{0_{\tilde{s}}}]\} + \{\,\}, \end{aligned} \tag{2.3}$$

where there will be in the second $\{\,\}$ on the right-hand side products of terms such as $[(1-C)\, g_{1_\eta}]_\eta$, $f_1 g_{1_\eta}$, etc.

The basis for the analysis resides in the solution to several problems each corresponding to $C \equiv 1$, $\sigma \equiv 1$, and $f \equiv f_0$, i.e. $f_1 \equiv 0$; in this case g_1 can be taken to be identically zero and the right-hand side of (2.3) is zero. The boundary and initial conditions of interest in the first problem to be considered are

$$g_0(\tilde{s}_i,\eta) = G_0(\eta), \quad g_0(\tilde{s},0) = g_{w\,0} = \text{const.}, \quad g(\tilde{s},\infty) = 1. \tag{2.4}$$

This problem represents a boundary layer, which has velocities described by the Blasius solution, which flows on a surface of constant enthalpy, and which has, at $\tilde{s} = \tilde{s}_i + 0$, an arbitrary distribution of stagnation enthalpy.

To find a solution let

$$g(\tilde{s},\eta) = g_{w,0} + (1-g_{w,0}) f_0' + g_{0,1}(\tilde{s},\eta), \tag{2.5}$$

and take

$$g_{0,1}(\tilde{s},\eta) = S_1(\tilde{s})\, N_1(\eta). \tag{2.6}$$

Substitution into (2.3) with a zero right-hand side yields

$$S_1 \sim \tilde{s}^{-\frac{1}{2}\lambda_1} \tag{2.7}$$

and

$$N_1'' + f_0 N_1' + \lambda_1 f_0' N_1 = 0, \tag{2.8}$$

where λ_1 is the separation parameter. The boundary conditions are

$$N_1(0) = N_1(\infty) = 0.$$

Clearly (2.8) defines eigenvalues λ_1 and eigenfunctions N_1. For the selection of λ_1 it is necessary to examine the asymptotic behaviour of N_1; for large η, (2.8) becomes

$$N_1'' + (\eta - \kappa)\, N_1' + \lambda_1 N_1 \simeq 0, \tag{2.9}$$

which is identical with the equation for H_1 discussed in the Appendix to Part 1 (Libby & Fox 1963). In particular for $\lambda_1 > 0$ all solutions yield $N_1(\infty) = 0$; however, if exponential behaviour is required, discrete values of $\lambda_1 > 0$ can be selected by numerical procedures as discussed below.

Consider some of the properties of these eigenfunctions; the usual procedures for establishing orthogonality can be readily applied to show that

$$\int_0^\infty (f_0'/f_0'')\, N_{1,m} N_{1,n}\, d\eta = C_{1,n}\, \delta_{mn}, \tag{2.10}$$

provided that $N_{1,m}$ decays exponentially for large η at least as fast as

$$\exp[-\tfrac{1}{4}(\eta-\kappa)^2].$$

Thus provided such asymptotic behaviour prevails the $N_{1,m}$ functions are orthogonal with respect to the weighting function (f_0'/f_0''). Also multiplication of (2.8) by $(N_1/f_0'')\,d\eta$ and integration over the infinite range of η leads readily to

$$\int_0^\infty (N_1'^2/f_0'')\,d\eta = \lambda_1 \int_0^\infty (f_0' N_1^2/f_0'')\,d\eta, \tag{2.11}$$

provided the aforementioned behaviour with $\eta \to \infty$ prevails; from (2.11) and the characteristics of f_0' and f_0'', it is clear $\lambda_1 > 0$. Similarly, consideration of (2.8) and of the equation for the complex conjugate of N_1 involving the complex conjugate of the eigenvalue λ_1 shows that λ_1 is only real. Finally, (2.8) can be put in Sturm–Liouville form so that the properties which result in (2.10) imply that the N_1 functions form a complete orthogonal set with respect to functions with exponential decay as $\eta \to \infty$.

Accordingly, numerical procedures can be employed to find the positive real values of λ_1, yielding exponential behaviour as $\eta \to \infty$. There is employed the approximate solution of (2.9) valid if $|1-\lambda_1|\,(\eta-\kappa)^{-2} \ll 1$, and if the power-law behaviour as $\eta \to \infty$ is suppressed; i.e.

$$N_1' \simeq -N_1(\eta-\kappa)\,[1+(1-\lambda_1)\,(\eta-\kappa)^{-2}]. \tag{2.12}$$

Thus, if at a value of η sufficiently large, a value of N_1 is assumed to prevail, then (2.12) provides N_1' and the integration can be started in the direction of η decreasing. At $\eta = 0$ the value of N_1 will in general be different from zero but its value provides a criterion for the selection of λ_1. After the eigenvalues have been selected, a final integration from $\eta = 0$ with $N_1'(0) = 1$ and with equal increments in η is carried out to $\eta = 6$ in order to obtain convenient tabulations of the eigenfunctions, and of a related function and to obtain the normalizing constant $C_{1,n}$.

The first ten eigenfunctions obtained according to this procedure are shown in figure 1. Presented below are the related eigenvalues and the normalizing constants

$$C_{1,n} \equiv \int_0^\infty (f_0'/f_0'')\,N_{1,n}^2\,d\eta.$$

n	1	2	3	4	5	6	7	8	9	10
$\lambda_{1,n}$	1·572	3·385	5·25	7·14	9·05	10·96	12·88	14·81	16·74	18·68
$C_{1,n}$	7·346	5·154	4·392	3·835	3·414	3·254	3·067	2·894	2·749	2·581

The eigenfunctions provide the solution for $g_{0,1}(\tilde{s},\eta)$ as

$$g_{0,1}(\tilde{s},\eta) = \sum_{n=1}^{\infty} A_{1,n}(\tilde{s}/\tilde{s}_i)^{-\frac{1}{2}\lambda_{1,n}} N_{1,n}(\eta), \tag{2.13}$$

where the $A_{1,n}$ coefficients are selected so that the initial conditions are satisfied; because of (2.4), (2.5) and (2.10),

$$A_{1,n} = \int_0^\infty (f_0'/f_0'')\,[G_0 - g_{w,0} - (1-g_{w,0})f_0']\,N_{1,n}\,d\eta/C_{1,n}. \tag{2.14}$$

It is noted that the solution given by (2.13) and (2.14) is exact in principle for general $G_0(\eta)$ but will be approximate in application if only a finite number of eigenfunctions are employed; of course, if the bracketed quantity in (2.14) is proportional to one of the available eigenfunctions or is represented by a finite sum of available eigenfunctions, then the solution is again exact.

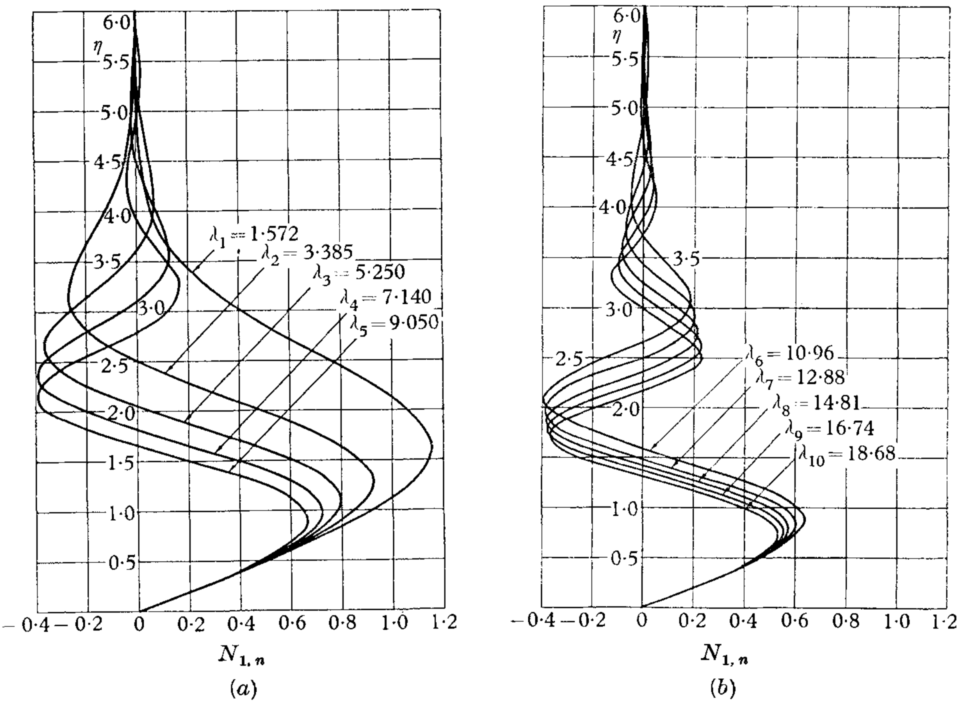

FIGURE 1. Eigenfunctions and eigenvalues for constant wall enthalpy: (*a*) $\lambda_{1,1}-\lambda_{1,5}$; (*b*) $\lambda_{1,6}-\lambda_{1,10}$.

A second problem, basic to the analysis, is given by (2.3) with its right-hand side identically zero and with initial and boundary conditions given by

$$g_0(\tilde{s}_i,\eta) = G_0(\eta), \quad g_{0_\eta}(\tilde{s},0) = 0, \quad g_0(\tilde{s},\infty) = 1. \tag{2.15}$$

This problem corresponds to a boundary layer which has velocities described by the Blasius solution, which flows on an adiabatic surface, and which has an arbitrary distribution of enthalpy at $\tilde{s} = \tilde{s}_i + 0$.† The solution may be found in much the same manner as that for $g_{0,1}$; let

$$g(\tilde{s},\eta) = 1+g_{0,2}(\tilde{s},\eta), \tag{2.16}$$

and

$$g_{0,2} = S_2(\tilde{s})\,N_2(\eta). \tag{2.17}$$

Then

$$S_2 \sim \tilde{s}^{-\frac{1}{2}\lambda_2}, \tag{2.18}$$

and

$$N_2''+f_0N_2'+\lambda_2 f_0' N_2 = 0, \tag{2.19}$$

† The solution of this problem can be applied directly to the determination of the concentration field either in terms of element mass fractions in reacting flows or in terms of the species mass fractions for non-reacting flows provided that all Lewis numbers are unity.

subject to the boundary conditions $N_2'(0) = N_2(\infty) = 0$. Again the λ_2's are separation constants which are eigenvalues of (2.19) and which may be selected so that exponential behaviour is obtained as $\eta \to \infty$.

The discussion pertaining to the characteristics of the $N_1(\eta)$ functions given previously apply as well to these $N_2(\eta)$ functions; i.e. $\lambda_2 > 0$ and real,

$$\int_0^\infty (f_0'/f_0'')\, N_{2,n} N_{2,m}\, d\eta = C_{2,n}\,\delta_{mn} \tag{2.20}$$

and the $N_{2,n}$ functions form a complete orthogonal set.

The solution for $g_{0,2}(\tilde{s}, \eta)$ is given by

$$g_{0,2}(\tilde{s}, \eta) = \sum_{n=1}^{\infty} A_{2,n} (\tilde{s}/\tilde{s}_i)^{-\frac{1}{2}\lambda_{2,n}} N_{2,n}(\eta), \tag{2.21}$$

where from (2.15), (2.16) and (2.20)

$$A_{2,n} = \int_0^\infty (f_0'/f_0'')\,(G_0 - 1)\, N_{2,n}\, d\eta / C_{2,n}. \tag{2.22}$$

For $\lambda_2 = 1$ the eigenfunction can be obtained in closed form by integration of (2.19);† there is obtained subject to the scaling condition $N_2(0) = 1$,

$$N_{2,1} = f_0''/f_0''(0). \tag{2.23}$$

Nine additional eigenvalues and eigenfunctions have been obtained by numerical integration with the same procedure as that employed for the selection of λ_1; they are shown in figure 2. The values of the eigenvalues and of the constants $C_{2,n}$ appearing in (2.20) are as given below:

n	1	2	3	4	5	6	7	8	9	10
$\lambda_{2,n}$	1	2·77	4·62	6·51	8·41	10·32	12·24	14·17	16·10	18·04
$C_{2,n}$	2·267	3·215	3·830	4·237	4·609	4·934	5·199	5·403	5·600	5·709

At this point in the presentation it is perhaps of interest to consider direct applications of these two sets of eigenvalues since they can be employed to provide exact solutions to certain problems.

The first set of eigenfunctions can be used directly to solve the problem of the heat transfer to a surface with a step-function distribution in wall enthalpy. Consider $C \equiv 1$, $\sigma \equiv 1$, $f \equiv f_0$ and let

$$g(\tilde{s}, 0) = \begin{cases} g_{w,1} & (0 \leqslant \tilde{s} \leqslant \tilde{s}_i), \\ g_{w,0} & (\tilde{s}_i \leqslant \tilde{s}), \end{cases} \tag{2.24}$$

where $g_{w,1}$, $g_{w,0}$ are constant. This problem is a classical one in boundary-layer theory; Tribus & Klein (1952) provide a convenient review of many problems connected with non-isothermal surfaces. Previous analyses of the energy equation in constant-pressure flows approximate the velocity field by $f \approx \eta^2$ and there-

† This case has been employed to validate the numerical procedures for the selection of the eigenvalues.

fore apply strictly only in the limit as $\sigma \to \infty$ (cf. Lighthill 1950). The solution given by (2.13) and (2.14) is in principle exact for $C \equiv 1$, $\sigma \equiv 1$, and may be obtained by letting

$$G_0 = g_{w,1} + (1 - g_{w,1}) f_0' \tag{2.25}$$

in (2.14), so that the coefficients $A_{1,n}$ can be evaluated once and for all in the form

$$\tilde{A}_{1,n} \equiv A_{1,n}/(g_{w,1} - g_{w,0}) = \int_0^\infty (f_0'/f_0'')(1 - f_0') N_{1,n} d\eta / C_{1,n}. \tag{2.26a}$$

Note that this integral can be evaluated in closed form as

$$\tilde{A}_{1,n} = [\lambda_{1,n} C_{1,n} f_0''(0)]^{-1}. \tag{2.26b}$$

Then the solution for g for $\tilde{s} \geqslant \tilde{s}_i$ is readily obtained from (2.5) and (2.13).

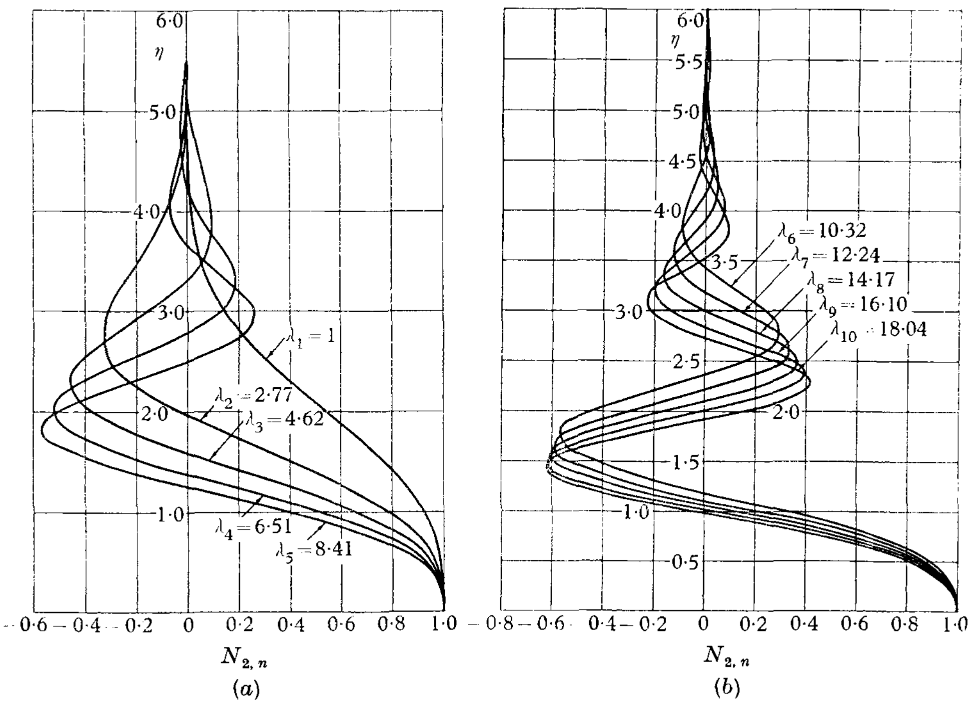

FIGURE 2. Eigenfunctions and eigenvalues for adiabatic wall: (*a*) $\lambda_{2,1} - \lambda_{2,5}$; (*b*) $\lambda_{2,6} - \lambda_{2,10}$.

When the first ten eigenfunctions available here are applied, the solution is of course again approximate; it is of interest to consider the profile development as shown in figure 3 for the special case of $g_{w,1} = 1$, i.e. an adiabatic surface followed by a cooled or heated wall. It is of interest to note in figure 3 that the representation of the initial profile, which in this case is constant, results in the classical 'overshoot'. Of interest in this problem is the heat transfer to the wall downstream of the discontinuity. For this special case of $g_{w,1} = 1$ and of two-dimensional flow† this can be shown to be

$$\tilde{N}_1 \equiv g_w'/[\rho_e u_e h_{s_e}(1 - g_{w,0}) N_{Rx}^{\frac{1}{2}}] = 2^{-\frac{1}{2}} f_{0_w}'' \left[1 + f_{0_w}''^{-1} \sum_{n=1}^{\infty} \tilde{A}_{1,n} (\tilde{s}/\tilde{s}_i)^{-\frac{1}{2}\lambda_{1,n}}\right]. \tag{2.27}$$

† Similar expressions involving as a parameter $(g_{w,1} - g_{w,0})/(1 - g_{w,0})$ can be readily derived for the general case.

Eckert (1950) obtains the following approximate result

$$\tilde{N}_1 \simeq 0{\cdot}332[1-(\tilde{s}/\tilde{s}_i)^{-\frac{3}{4}}]^{-\frac{1}{3}}, \tag{2.28}$$

which is functionally quite different. However, if the first ten eigenfunctions are used, the comparison shown in figure 4 is obtained.

Similarly an exact solution to a related problem can be obtained from the

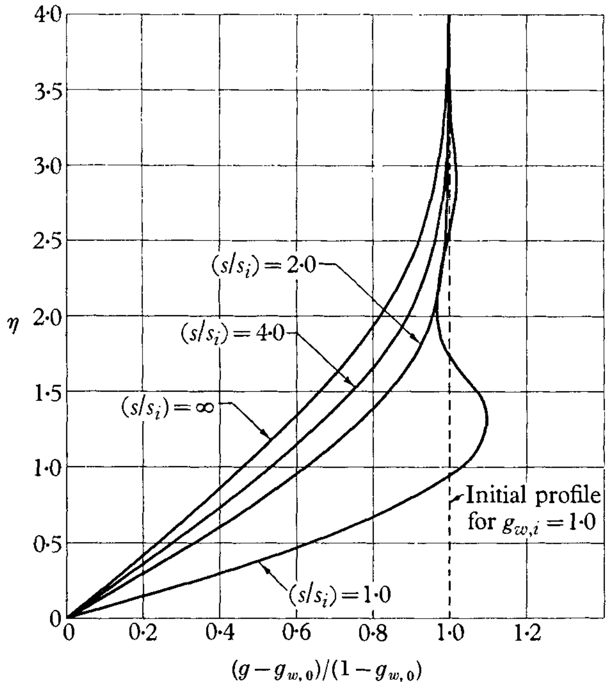

FIGURE 3. Enthalpy profiles for step-function wall enthalpy.

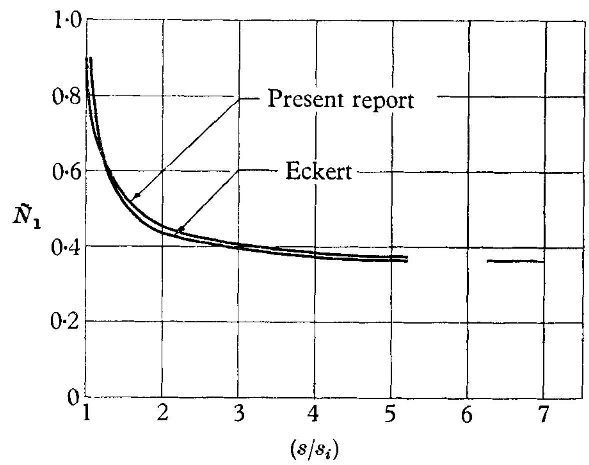

FIGURE 4. Distribution of heat transfer for step-function wall enthalpy. ———, Asymptotic values.

$N_{2,n}$ functions. Consider then a flow with $C \equiv 1$, $\sigma \equiv 1$, $f \equiv f_0$, and with wall boundary conditions given by

$$\left.\begin{aligned} g(\tilde{s}, 0) &= g_{w,1} \quad (0 \leqslant \tilde{s} \leqslant \tilde{s}_i), \\ g_\eta(\tilde{s}, 0) &= 0 \quad\quad (\tilde{s} > \tilde{s}_i). \end{aligned}\right\} \tag{2.29}$$

Such a flow involves a constant-temperature initial surface followed by an adiabatic downstream surface. This problem has also been treated in the past by approximate methods and by an adaptation of the analysis of Lighthill corresponding to $\sigma \to \infty$ (cf. Durgin 1959 for *inter alia* a review of the pertinent literatuıe).

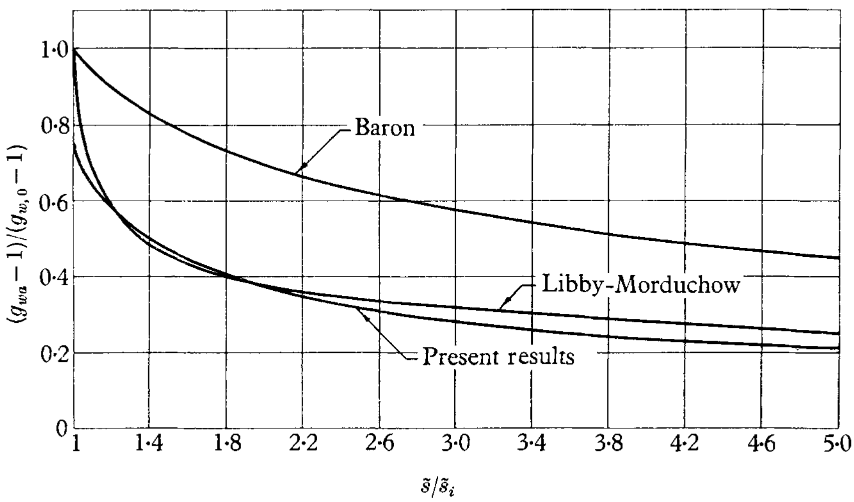

FIGURE 5. Distribution of adiabatic wall enthalpy downstream of a constant wall enthalpy section.

According to the present analysis G_0 in (2.22) is given by (2.25) so that the coefficients $A_{2,n}$ are given by

$$\tilde{A}_{2,n} \equiv A_{2,n}/(g_{w,0}-1) = \int_0^\infty (f_0'/f_0'')(1-f_0')\,N_{2,n}\,d\eta/C_{2,n}. \tag{2.30a}$$

Again, (2.30a) can be evaluated exactly as

$$\tilde{A}_{2,n} = [\lambda_{2,n} C_{2,n}]^{-1}. \tag{2.30b}$$

Of interest in this problem is the distribution of g_w on the adiabatic surface, i.e. for $\tilde{s} > \tilde{s}_i$; it is given from (2.16), (2.21), and (2.30) as

$$(g_{aw}-1)/(g_{w,0}-1) = \sum_{n=1}^{\infty} \tilde{A}_{2,n}(\tilde{s}/\tilde{s}_i)^{-\frac{1}{2}\lambda_{2,n}}. \tag{2.31}$$

The results obtained from the first ten eigenvalues, i.e. from those available, are presented in figure 5. Comparison is made to the results presented by Baron (1956) and Libby & Morduchow (1954).

Consider next as an application of the above analysis the problem treated according to the strip method by Pallone (1961) and by finite differences by

Howe (1959). This involves a porous surface with injection varying as $\tilde{s}^{-\frac{1}{2}}$ for $\tilde{s} < \tilde{s}_i$, followed by an adiabatic impermeable surface for $\tilde{s} > \tilde{s}_i$. It is desired to compute the distribution of adiabatic surface enthalpy on the downstream region. It has been assumed by Pallone and Howe that $C \equiv 1$ and that the Prandtl number is 0·72. An approximate solution to this problem can be obtained according to the analysis and point of view of the present report by considering the solutions to be given by

$$g(\tilde{s},\eta) = g_0(\tilde{s},\eta) + (1-\sigma)\, g_{1,1}(\tilde{s},\eta) + g_{1,2}(\tilde{s},\eta), \tag{2.32}$$

where g_0 satisfies the initial and boundary conditions and is given by (2.16), (2.21) and (2.22) with $G_0(\eta)$ obtained from Low (1955). This zero-order solution g_0 does not account either for the perturbation due to non-unity Prandtl number or for the perturbation to the velocity field due to the mass transfer. The first-order corrections therefore are given respectively by $g_{1,1}$ and $g_{1,2}$ which from (2.3) may be seen to be determined by

$$(g_{1,i})_{\eta\eta} + f_0(g_{1,i})_\eta - 2\tilde{s}f_0'(g_{1,i})_{\tilde{s}} = H_i(\tilde{s},\eta) \quad (i = 1, 2), \tag{2.33}$$

where
$$H_1 = f_0 g_{0_\eta} + 2\tilde{m}(f_0' f_0'')' - 2\tilde{s} f_0' g_{0_{\tilde{s}}}, \tag{2.34a}$$

and
$$H_2 = -f_1 g_{0_\eta} + 2\tilde{s}(f_{1_\eta} g_{0_{\tilde{s}}} - f_{1_{\tilde{s}}} g_{0_\eta}). \tag{2.34b}$$

The boundary and initial conditions on each correction function are homogeneous, i.e.

$$g_{1,i}(\tilde{s}_i,\eta) = (g_{1,i})_\eta(\tilde{s},0) = g_{1,i}(\tilde{s},\infty) = 0. \tag{2.35}$$

The solutions for the two functions $g_{1,i}$ $(i = 1, 2)$ are formally identical and may be obtained in terms of a Green's function constructed as follows: define $G(\tilde{s},\eta,\tilde{s}_0,\eta_0)$ by

$$G_{\eta\eta} + f_0 G_\eta - 2\tilde{s}f_0' G_{\tilde{s}} = \delta(\tilde{s}-\tilde{s}_0)\,\delta(\eta-\eta_0), \tag{2.36}$$

and let
$$G(\tilde{s},\eta,\tilde{s}_0,\eta_0) = \sum_{n=1}^{\infty} G_n(\tilde{s},\tilde{s}_0,\eta_0)\, N_{2,n}(\eta), \tag{2.37}$$

and
$$\delta(\eta-\eta_0) = \sum_{n=1}^{\infty} D_{2,n} f_0' N_{2,n}. \tag{2.38}$$

The coefficients $D_{2,n}$ involve η_0 as a parameter and because of the orthogonality of the $N_{2,n}$ functions, are given as

$$D_{2,n} = N_{2,n}(\eta_0)/f_0''(\eta_0)\, C_{2,n}. \tag{2.39}$$

Substitution of (2.37) and (2.38) into (2.36) and consideration of (2.19) defining the $N_{2,n}$ function results in

$$\lambda_{2,n} G_n + 2\tilde{s}G_n' = -D_{2,n}\,\delta(\tilde{s}-\tilde{s}_0), \tag{2.40}$$

where here $(\;)' \equiv d/d\tilde{s}$. It is convenient to let $G_n \equiv 0$ for $\tilde{s}_i \leqslant \tilde{s} < \tilde{s}_0$ so that for $\tilde{s} > \tilde{s}_0$ the solution of (2.40) is

$$G_n = -(D_{2,n}/2\tilde{s}_0)\,(\tilde{s}/\tilde{s}_0)^{-\frac{1}{2}\lambda_{2,n}}, \tag{2.41}$$

and the requisite Green's function is available. The solutions for $g_{1,i}$ are thus

$$g_{1,i}(\tilde{s},\eta) = \int_0^\infty \int_{\tilde{s}_i}^{\tilde{s}} G(\tilde{s},\eta,\tilde{s}_0,\eta_0)\, H_i(\tilde{s}_0,\eta_0)\, d\tilde{s}_0\, d\eta_0, \tag{2.42}$$

which must be evaluated numerically. In many applications properties at the surface $\eta = 0$ are of interest so that $\tilde{s}$ becomes the only parameter in the integration.

The results of this analysis have been compared to the calculations of Pallone (1961) for the case of $f_w = -0{\cdot}5$ in the upstream portion of length $x = L$ in a flow with a free-stream Mach number of 3·0. The variable $\tilde{s}/\tilde{s}_i$ can be written as x/L and the results of primary interest can be presented as in figure 6, i.e. in terms of

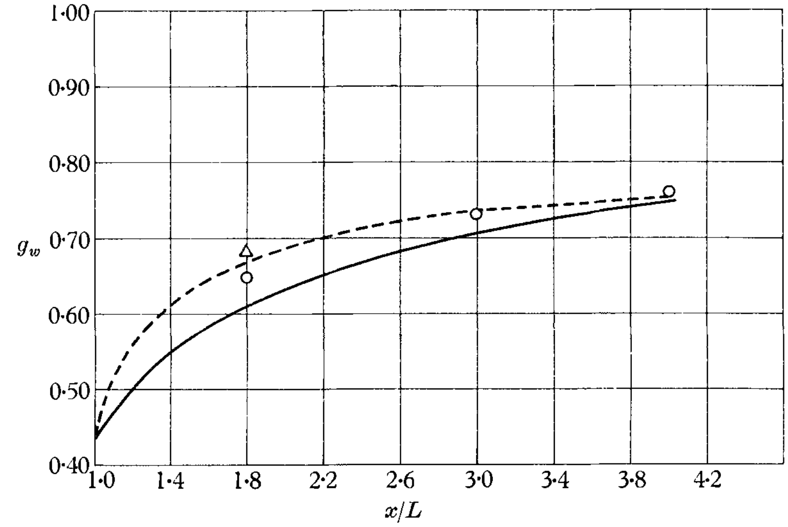

FIGURE 6. Distribution of adiabatic wall enthalpy downstream of a porously cooled section. ———, Present results, zero order; – – –, Pallone; ○, five terms; △, ten terms—present results with f_1 and $\sigma \neq 1$ included. ———, $f_w = 0{\cdot}5$; $M_\infty = 3{\cdot}0$.

g_w versus x/L. Shown there is the distribution of $g_0(\tilde{s}, 0)$ and for $x/L = 1{\cdot}8$, $3{\cdot}0$ and $4{\cdot}0$ the values of g_w, obtained from (2.32) with $\sigma = 0{\cdot}72$ and with $f_1(\tilde{s}, \eta)$ computed from the analysis in Part 1. There is demonstrated the use of five and ten eigenfunctions in the representation of the Green's functions. The numerical integration of (2.42) was carried out on a Bendix G-15 computer using a trapezoidal rule in both variables, $\tilde{s}_0$, η_0. The good agreement with the more accurate calculations of Pallone will be noted. The improvement, at $x/L = 1{\cdot}8$, with the ten eigenfunctions is noted; similar results can be expected for the other values of x/L.

The same approach can be used for other problems involving initial distributions of g; for example, if the effect on the adiabatic surface temperature of variable C is desired, it is possible to estimate C as a function of f_0 and g_0 and to add a correction function $g_{1,3}(\tilde{s}, \eta)$ on the right-hand side of (2.32). The forcing function $H_3(\tilde{s}, \eta)$ is in this case obtained from (2.3) as

$$H_3 = [(1-C)\, g_{0_\eta}]_\eta. \tag{2.43}$$

The solution given by (2.42) again applies. Similarly for problems involving initial enthalpy profiles and a specified wall enthalpy, $g_{w,0}$, the first-order corrections due to variations of C, to $\sigma \neq 1$, and to f_1, may be determined from a solution of the form (2.42) but with the Green's function expressed in terms of the $N_{1,n}(\eta)$ functions rather than of the $N_{2,n}(\eta)$ functions employed above.

In passing it is perhaps worth making several remarks; the second and higher approximations for $g_1(\tilde{s},\eta)$ can be obtained by application of (2.24) wherein the H_i forcing function is obtained from the appropriate additional terms on the right-hand side of (2.3) and is expressed in terms of previously obtained solutions for g_1. In problems involving $C \not\equiv 1$ the most accurate available representation of C can of course be incorporated in the H_i function. In obtaining these higher approximations the complete correction functions and certain derivatives thereof must be evaluated at a sufficient number of points $(\tilde{s},\eta)$ so that the integration for the next-order corrections in the form (2.42) can be carried out. Problems of heterogeneous boundary layers involving no gas-phase reaction but surface reaction of first order with a finite catalicity ζ require sets of eigenfunctions satisfying the condition $N'_{i,n}(0)-\zeta N_{i,n}(0) = 0$, $\zeta = \text{const.}$, but can be treated by the same techniques as employed here. Finally, problems of boundary layers with finite rate chemistry would appear to be tractable by iteration techniques based on (2.42); in this case the H_i function involves the creation terms and must be determined from a previous iterate.

3. Power-law distributions of wall enthalpy†

From the point of view demonstrated above consider the energy field in a boundary layer described by the Blasius solution with $C = 1$, $\sigma = 1$, subject to the conditions

$$g_0(0,\eta) = 1, \quad g_0(\tilde{s},0) = g_w(\tilde{s}), \quad g_0(\tilde{s},\infty) = 1, \tag{3.1}$$

where $0 \leqslant \tilde{s} \leqslant \tilde{s}_L$. This corresponds to the problem treated by Chapman & Rubesin (1949) but with $\sigma = 1$. Following their analysis, represent $g_w(\tilde{s})$ as

$$g_w(\tilde{s}) = 1+\sum_{n=0}^{\infty} D_{w,n}(\tilde{s}/\tilde{s}_L)^{\frac{1}{2}n} \quad (n = 0,2,4,6,\ldots), \tag{3.2}$$

where $D_{w,n}$ are known coefficients of a power series representation of $g_w(\tilde{s})$. Let

$$g_0(\tilde{s},\eta) = 1+\sum_{n=0}^{\infty} A_{3,n}(\tilde{s}/\tilde{s}_L)^{\frac{1}{2}n} N_{3,n}(\eta). \tag{3.3}$$

† It will be recognized from the above discussion that the solution to the step-function distribution of g_w given by (2.24)–(2.26) can be employed as a unit solution for the treatment of arbitrary distributions of g_w with $C \equiv 1$, $\sigma \equiv 1$. The results

$$g(\tilde{s},\eta) = g_w(\tilde{s})+[1-g_w(\tilde{s})]f_0' - \int_0^s \tilde{g}(\tilde{s},\eta,\xi)\,(dg_w/d\xi)\,d\xi,$$

where

$$\tilde{g}(\tilde{s},\eta,\xi) = \begin{cases} 0 & (\tilde{s} < \xi), \\ \sum_{n=1}^{\infty} \hat{A}_{1,n}(\tilde{s}/\xi)^{-\frac{1}{2}\lambda_{1,n}} N_{1,n}(\eta) & (\tilde{s} > \xi), \end{cases}$$

where $\hat{A}_{1,n} = \tilde{A}_{1,n}$ with $g_{w,1} = 1$, $g_{w,0} = 0$, and where the integral is taken in the Stieltjes sense. By application of the perturbation point of view this unit problem can be generalized for $f_1 \not\equiv 0$, $C \not\equiv 1$, $\sigma \not\equiv 1$. With a finite number of $N_{1,n}$ functions available this solution becomes approximate, whereas with the power law distributions discussed here the solutions for $C \equiv 1$, $\sigma \equiv 1$ are exact.

Then the equation for $N_{3,n}$ is obtained by substitution into (2.3) with zero right-hand side as

$$N''_{3,n} + f_0 N'_{3,n} - n f'_0 N_{3,n} = 0, \tag{3.4}$$

which may conveniently be subjected to the conditions $N_{3,n}(0) = 1$, $N_{3,n}(\infty) = 0$. The asymptotic behaviour of $N_{3,n}(\eta)$ is given by (2.12) with λ_1 replaced by $-n$; in this case the slope $N'_{3,n}(0)$ must be selected so that $\alpha_2 = 0$. Since (3.4) is linear, two integrations thereof with arbitrary but different values of $N'_{3,n}(0)$ can be

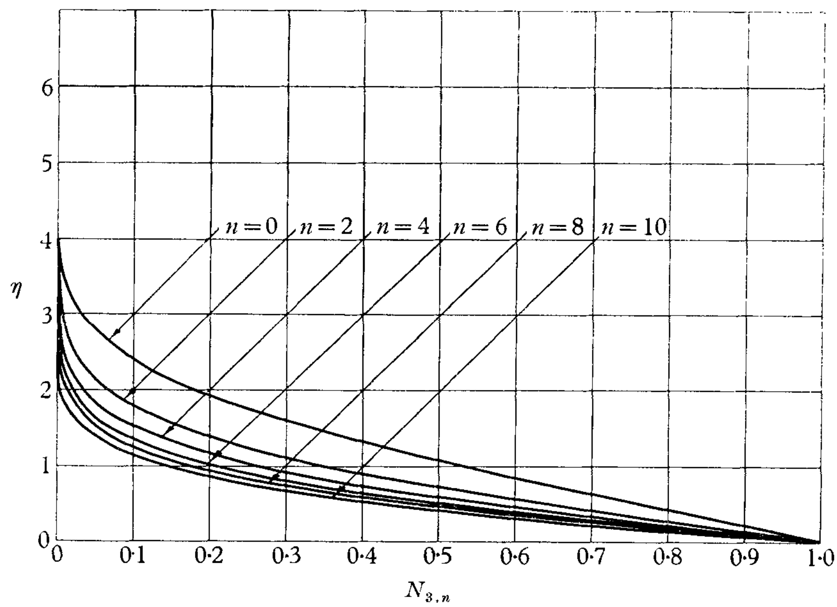

FIGURE 7. Characteristic functions for variable wall enthalpy. n: 0, 2, 4, 6, 8, 10; $-N'_{3,n}(0)$: 0·4696, 0·76714, 0·93309, 1·0553, 1·1543, 1·2393.

carried out to sufficiently large η so that the asymptotic solution is valid and then can be linearly combined so that $\alpha_2 = 0$ for the combination.†

Because of the boundary conditions on $N_{3,n}(\eta)$ it is clear from a comparison of (3.2) and (3.3) that $D_{w,n} = A_{3,n}$ so that (3.3) is the requisite solution. For $n = 0$, the solution of (3.4) satisfying the appropriate boundary conditions is

$$N_{3,0} = 1 - f'_0. \tag{3.5}$$

The solutions corresponding to $n = 2, 4, 6, 8, 10$ have been obtained numerically; they are shown in figure 7.

Consider now the influence of non-unity Prandtl number treated as a perturbation to the solution for g_0 given by (3.3); it is convenient to let

$$g(\bar{s},\eta) \simeq 1 + \sum_{n=0,2\ldots} A_{3,n}(\bar{s}/\bar{s}_L)^{\frac{1}{2}n}\,[N_{3,n} + (1-\sigma)\,N_{4,n}] + 2\tilde{m}(1-\sigma)\,N_5, \tag{3.6}$$

† An alternate procedure may be employed by initiating the calculation at sufficiently large η so that the asymptotic solution is valid and then with $\alpha_2 \equiv 0$ integrating to the wall. The solution then need only be scaled to obtain the desired behaviour of $N_{3,n}(0)$.

so that from (2.3) the two effects of $\sigma \neq 1$, i.e. that due to altered heat transfer and that due to viscous heating, are obtained separately. The equations for $N_{4,n}$ and N_5 are

$$N''_{4,n} + f_0 N'_{4,n} - n f'_0 N_{4,n} = f_0 N'_{3,n} - n f'_0 N_{3,n}, \tag{3.7}$$

$$N''_5 + f_0 N'_5 = (f'_0 f''_0)'. \tag{3.8}$$

The boundary conditions on $N_{4,n}$ and N_5 are homogeneous.

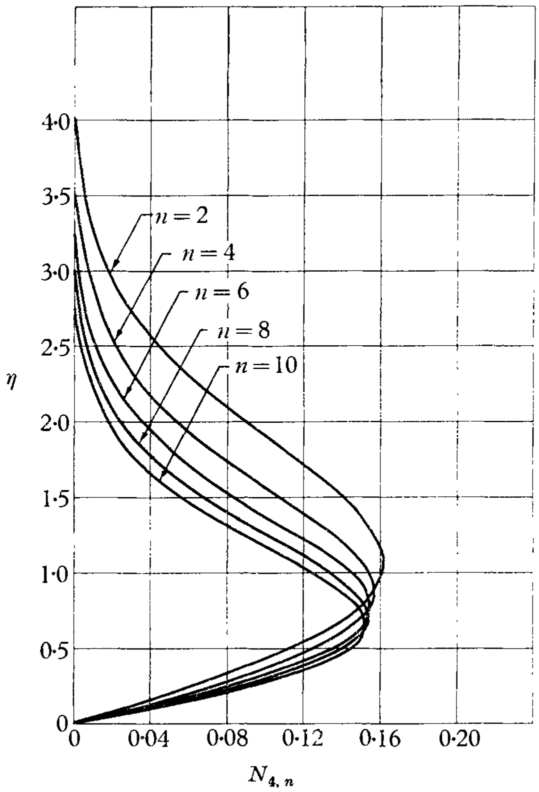

FIGURE 8. Characteristic functions for non-unity Prandtl number. n: 2, 4, 6, 8, 10; $N'_{4,n}(0)$: 0·260, 0·314, 0·354, 0·386, 0·413.

The solutions for $N_{4,0}$ and N_5 may be obtained in terms of the Blasius solution by quadrature; they are

$$N_{4,0} = c_1 f'_0 + f_0^2/2 + f'_0 \ln f''_0, \tag{3.9}$$

$$N_5 = (f'_0/4)(f'_0 - 1) + f_0 f''_0/2, \tag{3.10}$$

where
$$c_1 = -\lim_{\eta\to\infty} [\ln f''_0 + (f_0^2/2)] = 1{\cdot}117.$$

The solutions for $N_{4,n}$ (n = 2, 4, 6, 8, 10) have been obtained numerically and are given in figure 8.

The functions $N_{3,n} + (1-\sigma) N_{4,n}$ should be considered approximations to the functions obtained with a digital computer by Chapman & Rubesin for a specific value of σ, namely $\sigma = 0{\cdot}72$ and with an analog computer by Tifford & Chu (1953) for $\sigma = 0{\cdot}5, 1, 2$. A comparison of the present approximate results for the wall gradients, $N'_{3,n}(0)+(1-\sigma)N'_{4,n}(0)$, with the more accurate calculations is shown in figure 9. It will be noted that quite an accurate prediction of the effect of non-unity Prandtl number on these gradients is obtained by the present results.

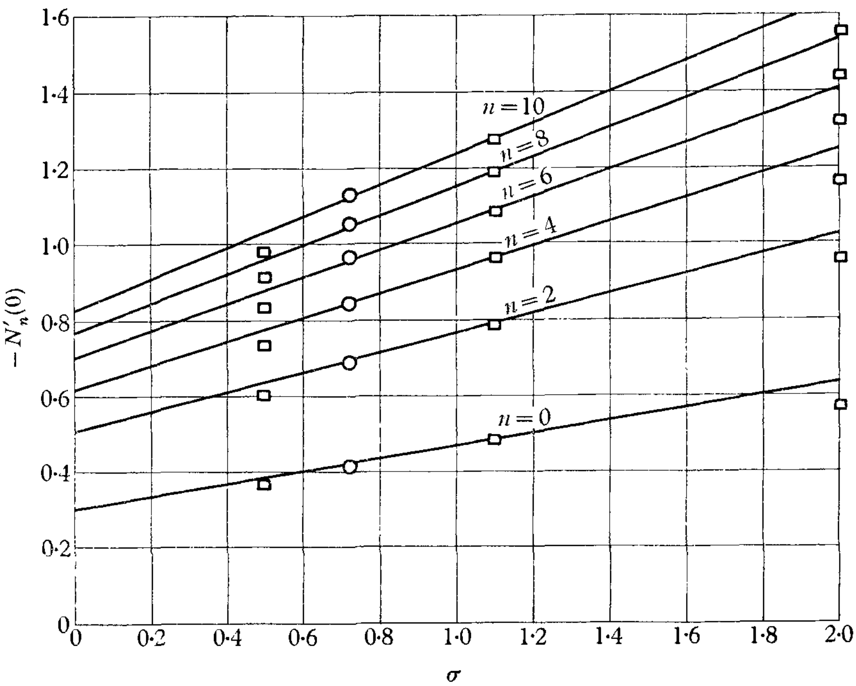

FIGURE 9. Effect of non-unity Prandtl numbers on wall gradients of unit solutions. ———, Present results $[N'_{3,n}+(1-\sigma)N'_{4,n}]_w$; ○, Chapman & Rubesin; □, Tifford & Chu.

It is perhaps of interest to note that the solution of the form (3.6) reduces in the special case of constant surface enthalpy $g_w = g_{w,0}$ to an approximate solution for $C \equiv 1$, $\sigma \neq 1$; in this case $A_{3,n} = 0$ $(n > 0)$, and $A_{3,0} = g_{w,0}-1$. The solution then becomes

$$g(\tilde{s},\eta) = g(\eta) = 1+(g_{w,0}-1)[(1-f'_0)+(1-\sigma)N_{4,0}]+2\tilde{m}(1-\sigma)N_5. \quad (3.11)$$

Of course, in this case an exact solution for g can be obtained by repeated quadrature (see for example, Schlichting 1955); however, (3.11) may be more convenient for rapid estimates of heat transfer. It is also noted that (3.11) yields an approximate recovery factor r defined in terms of the adiabatic wall enthalpy h_{aw} according to

$$h_{aw}/h_e = 1+r(u_e^2/2h_e). \quad (3.12)$$

If terms quadratic in $(1-\sigma)$ are neglected, (3.11) results in

$$r \simeq \tfrac{1}{2}(1+\sigma), \quad (3.13)$$

which is in agreement with the frequently employed approximation $r \simeq \sigma^{\frac{1}{2}}$ obtained by correlation of more accurate numerical calculations.

A further comparison can be made for the numerical example discussed by Chapman & Rubesin; although their surface temperature is expressed as a distribution of $(T_w - T_{aw})/T_{aw}$ the corresponding distribution of g_w in terms of the parameters of the present report is found to be

$$g_w(\bar{x}) = [1 + (\tilde{m}/2)(\sigma - 1)][1 + 0\cdot25 - 0\cdot83\bar{x} + 0\cdot33\bar{x}^2], \tag{3.14}$$

where the approximation (3.13) has been employed in (3.12), where

$$T_{aw}/T_e = h_{aw}/h_e,$$

and where $\bar{x} = \tilde{s}/\tilde{s}_L$. Clearly in this example, $n = 0, 2, 4$ only. It is convenient to introduce the adiabatic value of g, i.e. the value of g corresponding to h_{aw}/h_e as given by (3.12) and (3.13); it is

$$g_{aw} = 1 + \tfrac{1}{2}\tilde{m}(\sigma - 1), \tag{3.15}$$

so that comparison of (3.14) and (3.2) implies

$$\left.\begin{aligned} D_{w,0} &= 1\cdot25g_{aw} - 1 = \tfrac{1}{2}\tilde{m}(\sigma-1) + 0\cdot25g_{aw}, \\ D_{w,2} &= -0\cdot83g_{aw}, \quad D_{w,4} = 0\cdot33g_{aw}. \end{aligned}\right\} \tag{3.16}$$

The comparison is made in terms of a Nusselt–Reynolds number denoted here as $\tilde{N}_2$ and defined as

$$\tilde{N}_2 \equiv (g'_w c_{p,e} x)/(k_e h_{aw} N_{Rx}^{\frac{1}{2}}). \tag{3.17}$$

This parameter is identical to the parameter of Chapman & Rubesin in the case of constant c_p. The comparison is shown in figure 10 for $\sigma = 0\cdot72$; also show is the heat transfer for $\sigma = 1$. The good agreement for the former value of the Prandtl number will be noted.†

As a final example of the treatment of the energy equation by the perturbation point of view, consider the following problem: Let $C \equiv 1$, $\sigma \equiv 1$, and seek a solution of the energy equation subject to the conditions

$$g(\tilde{s}_i, \eta) = G_0(\eta), \quad g(\tilde{s}, \infty) = 1, \quad g(\tilde{s}, 0) = g_w(\tilde{s}). \tag{3.18}$$

This will be recognized as a combination of the initial-value problem of §2 and the specified wall-enthalpy problem of this section. In this case, however, it is permissible, and indeed may be more convenient, to consider negative integer values in the power-series representation of $g_w(\tilde{s})$‡. In order to emphasize this difference, the open range of $\tilde{s}$, $\tilde{s} \geqslant \tilde{s}_i > 0$ will be considered and it will be assumed that

$$g_w(\tilde{s}) = g_{w,\infty} + \sum_{n=2}^{\infty} D_{w,n}(\tilde{s}/\tilde{s}_i)^{-\frac{1}{2}n}, \quad (n = 2, 4, 6, \ldots), \tag{3.19}$$

where $g_{w,\infty}$ is a constant and where the $D_{w,n}$ coefficients are known.

It will be convenient to seek a solution in the form

$$g(\tilde{s}, \eta) = g_{w,\infty} + (1 - g_{w,\infty})f'_0 + \sum_{n=1}^{\infty} A_{4,n}(\tilde{s}/\tilde{s}_i)^{-\frac{1}{2}\lambda_{1,n}} N_{1,n}(\eta) + \sum_{n=2}^{\infty} A_{5,n}(\tilde{s}/\tilde{s})^{-\frac{1}{2}n} N_{3,n}(\eta), \tag{3.20}$$

† If the second form for $D_{w,0}$ above is employed, then the quantity g_{aw} factors from the expression for g_w, provided a second-order term proportional to $(\sigma-1)^2$ is neglected; in this case the Mach number does not have to be specified.

‡ The authors are indebted to Mr Paul Taub for pointing this out.

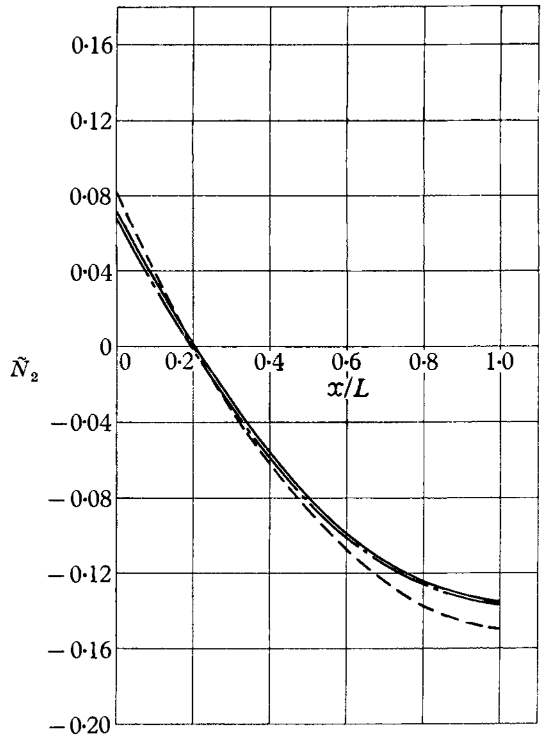

FIGURE 10. Variation of heat transfer rate. – – –, Present report, $\sigma = 1{\cdot}0$; ——, present report, $\sigma = 0{\cdot}72$; —·—, Chapman & Rubesin, $\sigma = 0{\cdot}72$.

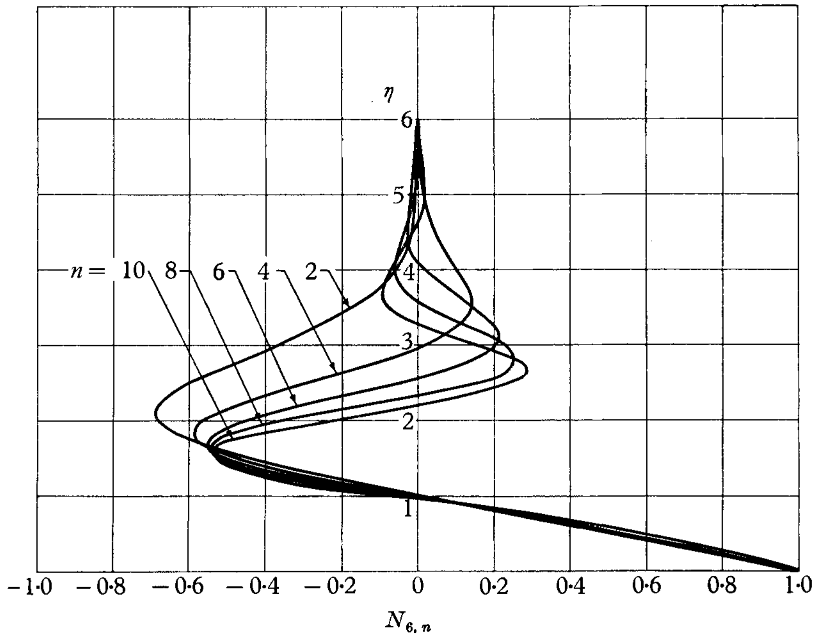

FIGURE 11. Characteristic functions for variable wall enthalpy. n: 2, 4, 6, 8, 10; $-N'_{6,n}(0)$: 0·95, 0·889, 0·816, 0·734, 0·642.

where $N_{1,n}(\eta)$ are the eigenfunctions presented here, and $N_{6,n}(\eta)$ are given by (3.4) with $n < 0$ and $N_{3,n} \to N_{6,n}$. If $N_{6,n}(0) = 1$, then comparison of (3.19) and (3.20) yields $D_{w,n} = A_{5,n}$. The initial conditions are satisfied by selecting $A_{4,n}$ so that

$$A_{4,n} = \int_0^\infty (f_0'/f_0'') \left[G_0 - g_{w,\infty} - (1-g_{w,\infty})f_0' - \sum_{n=2}^{\infty} D_{w,n} N_{6,n}(\eta)\right] N_{1,n}\, d\eta / C_{1,n}. \tag{3.21}$$

The functions $N_{6,n}(\eta)$ for $n = 2, 4, 6, 8, 10$ have been found numerically; they are shown in figure 11. Clearly the techniques, previously described for taking into account the effects of variable C and of $\sigma \neq 1$, can be applied to this problem.

4. Concluding remarks

There has been presented a treatment of problems involving arbitrary initial energy profiles and power-law distributions of wall enthalpy. Exact solutions to several problems involving constant $\rho\mu$, unity Prandtl number and velocity fields given by the Blasius solutions have been obtained in terms of complete orthogonal sets of functions which appear to provide convenient approximate solutions to other problems in boundary-layer theory. In particular the effects of variability of the product $(\rho\mu)$, of non-unity but constant Prandtl number, and of deviations of the velocity field from that given by the Blasius solution have been treated as perturbations. Because of the character of the sets of functions higher-order effects can be computed systematically by quadrature. The close analogy between the equations for energy conservation, for species conservation with no gas-phase reaction and for element conservation indicates the applicability of this analysis to a variety of problems related to heterogeneous and reacting flows.

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REFERENCES

Baron, J. R. 1956 The binary-mixture boundary layer associated with mass transfer cooling at high speeds. Massachusetts Institute of Technology, Naval Supersonic Lab. TR 160.

Chapman, D. R. & Rubesin, M. W. 1949 Temperature and velocity profiles in the compressible laminar boundary layer with arbitrary distribution of surface temperature. *J. Aero. Sci.* **16**, 547.

Durgin, F. H. 1959 An experiment on the insulating properties of boundary layers. Massachusetts Institute of Technology DSR No. 7509, AFOSR TN 57-392, AD 132467.

Eckert, E. R. G. 1950 *Introduction to the Transfer of Heat and Mass*, 1st Ed. New York: McGraw-Hill Book Company, Inc.

Hayes, W. D. & Probstein, R. F. 1959 *Hypersonic Flow Theory, Applied Mathematics and Mechanics*, vol. 5. New York: Academic Press.

HOWE, J. T. 1959 Some finite difference solutions of the laminar compressible boundary layer showing the effects of upstream transpiration cooling. *NASA Memo*, no. 2-26-59A.

LEES, L. 1956 Laminar heat transfer over blunt-nosed bodies at hypersonic speeds. *Jet Propulsion*, **26**, 259.

LIBBY, P. A. & FOX, H. 1963 Some perturbation solutions in laminar boundary layer theory. Part 1. The momentum equation. *J. Fluid Mech.* **17**, 433.

LIBBY, P. A. & MORDUCHOW, M. 1954 Method for calculation of compressible laminar boundary layer with axial pressure gradient and heat transfer. *NACA TN*, no. 3157.

LIGHTHILL, M. J. 1950 Contributions to the theory of heat transfer through a laminar boundary layer. *Proc. Roy. Soc.* A, **202**, 359.

LOW, G. M. 1955 The compressible laminar boundary layer with fluid injection. *NACA TN*, no. 3404.

PALLONE, A. 1961 Non-similar solution of the compressible-laminar-boundary-layer equations with application to the upstream-transpiration cooling problem. *J. Aero. Sci.* **28**, 449.

SCHLICHTING, H. 1955 *Boundary Layer Theory*, pp. 264–268. New York: McGraw-Hill Book Company, Inc.

TIFFORD, A. N. & CHU, S. T. 1953 Heat transfer and frictional effects in laminar boundary layers. Part I. Wedge Analyses. *WADC TR* no. 53-288.

TRIBUS, M. & KLEIN, J. 1952 Forced convection from nonisothermal surfaces. *Heat Transfer*, pp. 211–235. Engineering Research Institute, University of Michigan.